15694

ARPA Network Information Center
Stanford Research Institute
Menlo Park, California 94025
NIC # 15694
Packet Radio Temporary Note 47

Network Analysis Corp.
14 March 1973

COMBINATORIAL ASPECTS OF MESSAGE FLOW IN PACKET RADIO NETS

PART II

Contents

I. Introduction

II. Outline of Problems

III. Message Distribution with Type I Slotting

IV. Message Distribution with Type II Slotting

V. Survival Probabilities of Messages

VI. Distribution of Arrivals and Receptions

VII. Remarks on the Closed Boundary Problem

VIII. Some Useful Inequalities

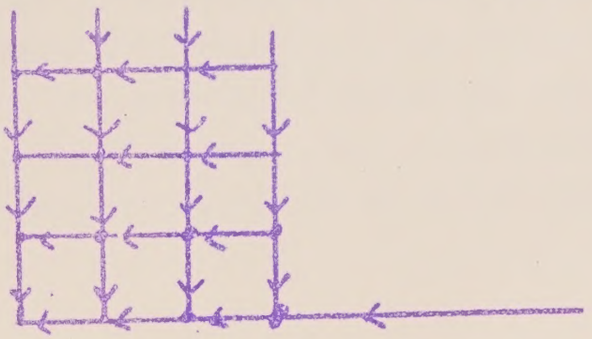
IX. Summary and Conclusions

X. Computer Analysis: A Closed Boundary Situation

I. INTRODUCTION:

This report is a continuation of "Combinatorial Aspects of Message Flow in Packet Radio Nets, Part I". In this analysis we focus on a particular model which was previously analyzed, and answer more detailed questions. We consider in a detailed way the problem of message distribution when all messages are transmitted in the direction of the origin. The model is essentially the same as studied in reference 1 except that some messages which are received at a repeater may not be repeated. When the message is not repeated we say it was not received. Thereby, we draw the distinction between arrivals and receptions.

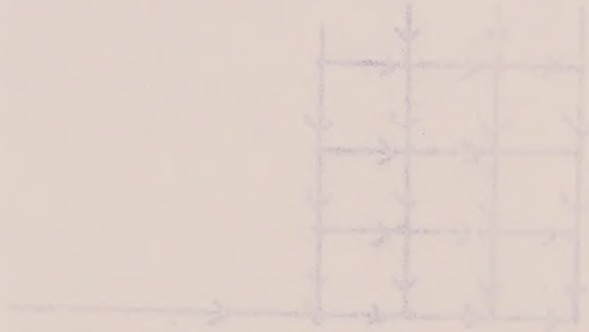
Pictorially, the model can be described as a set of nodes or vertices which represent repeaters. The repeaters are at the integral lattice points of the plane, and the origin is considered the fixed station. Messages are repeated only in the direction of the origin. We can examine only the first quadrant due to symmetry. The arrows represent possible directions for a message.



A repeater at distance d transmits to those repeaters one unit away which are at distance d-1 from the origin.

OUR BASIC ASSUMPTIONS:

A. Starting at time t=0, and at quantized time periods afterwards (perhaps 1 sec.) t= 1, 2,..., messages originate at each repeater independently according to a Poisson probability law. That is, the probability that exactly k



messages originate is $\frac{e^{-\lambda}\lambda^{k}}{k!}$ when k = 0, 1, 2,... where λ is a constant and is the mean or average number of originations.

B. Messages which are "received" at any repeaters are instantly repeated to all repeaters one unit closer to the fixed station. They arrive one time unit later at the neighboring repeaters.

C. A repeater has the capacity to "receive" at most m-messages. As each message arrives at a repeater it is randomly and independently assigned to a "slot"; there are m-slots. Two possible modes of operation are considered. In the first mode a message is "received" if it is the only message in its slot. In the second mode the number of messages "received" is the same as the number of non-empty slots.

II. OUTLINE OF PROBLEMS:

We shall describe in this section the specific questions which are of interest in this model.

PROBLEM 1:

Find $P_{kj}(j = 0, 1, 2,\ldots,k)$, $k = 0, 1, 2,\ldots$, which is the probability that exactly j messages are received given that k arrive at a given repeater when type 1 "slotting" is used. That is, a message is received if and only if it is the only message assigned to its slot. In type 1 slotting, the number of messages received is given by the number of slots which have exactly one message. Note that P_{kj} is independent of time. Given that k messages arrive, the distribution of the number received does not depend on time or what happens at neighboring repeaters.

PROBLEM 2:

Find P^{*}_{kj} $(j = 0, 1, 2,\ldots,k)$, $k = 0, 1, 2,\ldots$, which is the probability that exactly j messages are received given that k arrive at a given repeater, when type II "slotting" is used. In type II slotting, the number of messages received is given by the number of non-empty slots. The same remarks concerning independence hold for this problem as discussed above for problem I.

PROBLEM 3:

Find the probability $P_d(t)$ that a message which originates at time t, at distance d from the origin, is received at the origin at time t+d. The probability $P_d(t)$ is called the survival probability, and should be computed under type I, and type II slotting.

PROBLEM 4:

Let $X_o^R(t)$ be the number of messages received at the origin at time t, and $X_o(t)$ be the number of messages which arrive at the origin at time t.

Let $X_{j,d}^R(t)$ be the number of messages which are "received" at a repeater with coordinates (j,d) at time t, and $X_{j,d}(t)$ be the number which arrive, $j = 0, 1, 2, \ldots, d$, $d = 1, 2, \ldots$

All X's are random variables; compute their expectations and or distributions under type I and type II slotting.

PROBLEM 5:

Solve problems 3 and 4 when the region containing the repeaters is closed at say distance B from the origin.

III. MESSAGE DISTRIBUTION WITH TYPE I SLOTTING:

We derive formulae for P_{kj}, the probability that j messages are "received" given that k messages "arrive" at a repeater. Each message is assigned independently and at random to one of m-identical slots. The number of "received" messages is given by the number of slots with exactly one message.

To solve this problem, we consider the "ball in cell" model of placing k distinct balls into m-distinct cells. The quantity P_{kj} is the probability that exactly j cells have exactly one ball in each.

Let S be the set of all k-tuples where each component is one of the integers 1, 2,...,m, i.e., $S = \{(a_1, a_2, \ldots, a_k): a_i \in \{1, 2, \ldots, m\}, i = 1, 2, \ldots, k$. Let A_ν be the event(subset) where cell ν has exactly one ball, i.e. the subset of S of those sequences where the integer ν appears exactly once.

$$\text{Let } \sum_{i_1<i_2<i_3<\ldots<i_w} P_{i_1,i_2,\ldots,i_w} = S_w$$

where $P_{i_1,i_2,\ldots,i_w}$ is the probability $P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_w})$

Digitized by the Internet Archive
in 2025 with funding from
Amateur Radio Digital Communications, Grant 151

https://archive.org/details/temporary-note-047

and the sum is over all subsets of integers of size w selected from [1, 2,...,m].

It is easy to compute S_w,

$$S_w = \sum_{i_1<i_2<\ldots<i_k} \frac{k(k-1)\ldots(k-w+1)}{m^k (m-w)^{w-k}} = \binom{m}{w} \frac{k!\,(m-w)^{k-w}}{m^k (k-w)!}$$

for $w = 1, 2,\ldots,\min(k,m)$, $S_w = 0$ for $w > \min(k,m)$.

By the well known variation of inclusion-exclusion, (see, e.g. reference 2, page 96),

$$P_{kj} = \sum_{\tau=0}^{m-j} (-1)^\tau \binom{j+\tau}{j} S_{j+\tau} ,$$

$$= \frac{i}{m^k} \sum_{\tau=0}^{m-j} (-1)^\tau \frac{\binom{j+\tau}{j}\binom{m}{j+\tau} k!\,(m-j-\tau)^{k-j-\tau}}{(k-j-\tau)!}$$

$$= \frac{\binom{m}{j}}{m^k} \sum_{\nu=0}^{m-j} (-1)^\nu \binom{m-j}{\nu} \frac{k!}{(k-j-\nu)!} (m-j-\nu)^{k-j-\nu} ,$$

When $j \le \min(k,m)$. Actually, P_{kj} depends on m and should be written as $P_{kj}(m)$. Von-Mises has shown that when m is large, $P_{kj}(m)$ can be approximated by;

$$P_{kj}(m) = \frac{e^{-\lambda_1}(\lambda_1)^j}{j!}$$ a Poisson variate with $\lambda_1 = ke^{\frac{-k}{m}}$.

This is an approximation for the binomial with $p = e^{\frac{-k}{m}}$. $q = 1-e^{\frac{-k}{m}}$ when k is large, p small and kp moderate.

$$P_{kj} = \binom{k}{j} e^{\frac{-jk}{m}} \left(1-e^{\frac{-k}{m}}\right)^{k-j} .$$

IV. MESSAGE DISTRIBUTION WITH TYPE II SLOTTING:

We derive formulae for P_{kj}^{*}, the probability that j messages are "received" given that k messages "arrive" at a repeater. Each message is assigned independently and at random to one of m-slots. The number of "received" messages is given by the number of slots with at least one message.

Again we use a "ball in cell" model and ask for the probability that exactly j boxes are not empty when we place k distinct balls in m-distinct boxes.

Using the same method of inclusion-exclusion as in problem 1, we find that;

$$P_{kj}^{*} = \binom{m}{j} \sum_{\nu=0}^{j} (-1)^{\nu} \binom{j}{\nu} \left(\frac{j-\nu}{m}\right)^{k}.$$

We can approximate P_{kj}^{*} by the same method as Section III, (see ref. 2, page 94)

$$P_{kj}^{*} \sim \frac{e^{-\lambda_2} \cdot \lambda_2^{m-j}}{(m-j)!} \qquad 0 \le j \le \min(k,m)$$

where λ_2 is $e^{-\frac{k}{m}}$ when m,k are large.

V. SURVIVAL PROBABILITIES OF MESSAGES:

Problem number three calls for the solution to the problem of finding the survival probability for a message which originates at a repeater with coordinates (j,d) at time t. We must first compute the probability that it is received at its original node. Let P_1^{*} be the probability that a given message which arrives at a repeater is received at that repeater. We must compute P^{*} under two nodes on types of slotting. We call P_1^{*}, P_2^{*} the values of P^{*} under type I and II slotting respectively.

Let S_k be the set of (k+1) tuples formed from the integers 1, 2,...,m. Clearly, $n(S_k) = m^{k+1}$, i.e., the number of elements in S_k is m^{k+1}. Let $P^*_{k,1}$ be the probability that the message is received given that k-other messages arrive and type I slotting is used. Clearly, $P^*_{0,1} = 1$, and for $k > 0$,

$$P^*_{k,1} = \frac{m(\text{number of elements in } S_k \text{ with } r \text{ in the } j^{th} \text{ position only})}{m^{k+1}}$$

$$P^*_{k,1} = \frac{m}{m^{k+1}}\left[(m-1)^k\right] = \frac{(m-1)^k}{m^k} = \left(1-\frac{1}{m}\right)^k$$

which is of course independent of time.

However, P^*_1 is dependent on time since it depends on the probability that k-other messages arrive: Therefore,

$$(5.1)\quad P^*_1(t) = \sum_{k=1}^{\infty} P^*_{k,1} \cdot P\{\text{exactly k other messages arrive at time t}\}.$$

In principle $P^*_1(t)$ is computable from the distribution of arrivals at a given node at time t.

We must compute $P^*_2(t)$ for type II slotting. In this system, the given message is received if it is selected at random from the messages in its slot.

$$P^*_2(t) = \sum_{k=0}^{\infty} P[\text{Reception/k arrivals at t}]\, P[\text{k arrivals at t}]$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} P[\text{Reception/k arrivals at t, j in same slot}]$$

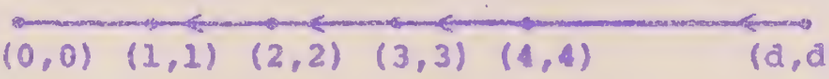
$$P\{\text{j in same slot/k arrival}] \cdot P[\text{k arrivals at t}].$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{1}{j+1} \cdot P\{\text{j in same slot/k arrivals}\} \cdot P\{\text{k arrivals at t}\}.$$

$$(5.2)\quad P_2^*(t) = \sum_{k=0}^{\infty} \frac{m}{k+1}\left[1-\left(1-\frac{1}{m}\right)^{k+1}\right] \cdot P[k \text{ arrivals at time } t].$$

Equations 5.1 and 5.2 can be solved once the distribution of message arrivals is determined.

To compute the solution to the problem of survival, let us, for simplicity, assume that the message arrives at a repeater with coordinates (d,d). Pictorially; the same problem for a repeater with coordinates (0,d).



Let $P_1(d,t)$ be the probability that a given message which originates at (d,d) at time t is received at the fixed station at (0,0) at time (t+d). Since the repeater at d-1 cannot "tell the difference" between a message arriving from (d,d) or originating, we can write:

$$P_1(d,t) = P_1(d-1,t+1)\ \ P[\text{message is initially received at } (d,d)].$$

The quantity, P[message is initially received at (d,d) was computed as (5.1) and (5.2) in the previous analysis for type I and type II slotting. Thus, the survival probabilities satisfy the fundamental difference equations for $P_1(d,t)$ and $P_2(d,t)$ the survival probabilities for type I and type II slotting respectively.

FUNDAMENTAL DIFFERENCE EQUATIONS:

$$(5.3)\quad P_1(d,t) = P_1(d-1,t+1) \sum_{k=0}^{\infty} \left(1-\frac{1}{m}\right)^k P[X_d(t) = k]:\ t \geq 0 \quad d \geq 1$$

with the initial condition,

$$(5.3A)\quad P_1(0,t) = \sum_{k=0}^{\infty} \left(1-\frac{1}{m}\right)^k P[X_d(t) = k].$$

$$(5.4)\quad P_2(d,t) = P_2(d-1,t+1)\cdot \sum_{k=0}^{\infty} \frac{m}{k+1}\left[1-\left(1-\frac{1}{m}\right)^{k+1}\right] P[X_d(t)=k]$$

for $t \geq 0 \quad d \geq 1$,

with the initial condition,

$$(5.4A)\quad P_2(0,t) = \sum_{k=0}^{\infty} \frac{m}{k+1}\left[1-\left(1-\frac{1}{m}\right)^{k+1}\right] P[X_{0,0}(t)=k].$$

For t=0 all distributions $X_{j,d}(0)$ are Poisson with mean λ. Thus, we have theoretically solved the survival probability problem in terms of the message arrived distributions. We now turn our attention to the solution of problem 4.

VI. DISTRIBUTION OF ARRIVALS AND RECEPTIONS:

We now discuss the difficult problem of finding the distribution of arrivals and receptions at each repeater at each point in time. In sections III and IV we have established formulae for P_{kj} which connect the arrival and reception distributions. It is possible to greatly simplify the problem by proving that there are only three random processes to determine.

A. $X_0(t)$ = arrival at origin at time t.

B. $X_d(t)$ = arrivals at distance d from the origin on an axis at time t.

C. $X(t)$ = arrivals off the axis at time t.

The corresponding quantities $X_0^R(t)$, $X_d^R(t)$, $X^R(t)$ denote the reception distributions which can be obtained from the arrival distributions and P_{kj}. These remarks are justified by the following theorem.

THEOREM: A. $P[X_{0,d}(t)=k] = P[X_{0,e}(t)=k]$ for each $d,e \geq 0$, each $t \geq 0$ and $k=0, 1, 2,\ldots$

B. $P[X_{j,d}(t)=k] = P[X_{\nu,w}(t)=k]$ for $k=0, 1, 2,\ldots$ for all pairs $j,\nu > 0, d,w > 0$.

C. $P[X^R_{0,d}(t)=k] = P[X^R_{0,\ell}(t)=k]$ for $k=0, 1, 2,\ldots$ for all pairs $d,\ell \geq 0$ and every t .

D. $P[X^R_{j,d}(t)=k] = P[X^R_{\nu,w}(t)=k]$ for $k=0, 1, 2,\ldots$ and all pairs $d,w > 0$ and $j,\nu > 0$.

PROOF: The proof follows by induction on t based on the facts that all repeaters start off with the identical Poisson process, the symmetry due to an unbounded region and the relationships between $X_{j,d}(t)$ and $X^R_{j,d}(t)$.

Thus we need "only" to determine the distributions of the random process $X_0(t)$, $X_d(t)$ and $X(t)$; $t \geq 0$. The initial conditions are given by,

$$P[X_0(0)=k] = P[X_d(0)=k] = P[X(0)=k] = \frac{e^{-\lambda}\lambda^k}{k!} \quad k=0, 1, 2, \ldots$$

For the "received" randon process,

$$P[X^R_0(0)=j] = P[X^R_d(0)=j] = P[X^R(t)=j]$$

$$= \sum_{k=0}^{\infty} P[X^R(0)=j \mid X(0)=k] \cdot P[X(0)=k]$$

$$= \sum_{k=j}^{\infty} P_{kj}\, P[X(0)=k]$$

$$= \sum_{k=j}^{\infty} P_{kj}\, \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=j}^{\infty} \frac{P_{kj}\lambda^k}{k!}$$

for type I slotting, and,

$$= e^{-\lambda} \sum_{k=j}^{\infty} \frac{P^*_{kj}\lambda^k}{k!} \quad \text{for type II slotting.}$$

We can substitute the formulae for P_{kj} and P^*_{kj} from sections III and IV into this equation, but the results are quite complicated.

The processes $X(t)$, $X_d(t)$ and $X_0(t)$ are each sequences of independent random variables over time, however, for each repeater, they depend on the values of earlier times at neighboring repeaters.

In fact, we can write the recursive equations,

(6.1) $X_0(t) = Y_1(t-1) + Y_2(t-1) + Y_3(t-1) + Y_4(t-1) + Y_5$

Where Y_1, Y_2, Y_3, Y_4 are identically distributed (not independent) random variables with the same distribution as $X^R_1(t-1)$ and Y_5 is a Poisson random variable which is independent of Y_1, Y_2, Y_3, Y_4. It

follows that,

$$(6.1A)\quad E[X_0(t)] = 4E[X_1^R(t-1)] + \lambda$$

Where E denotes expectation.

Similarly,

$$(6.2)\quad X(t) = W_1(t-1) + W_2(t-1) + W_3$$

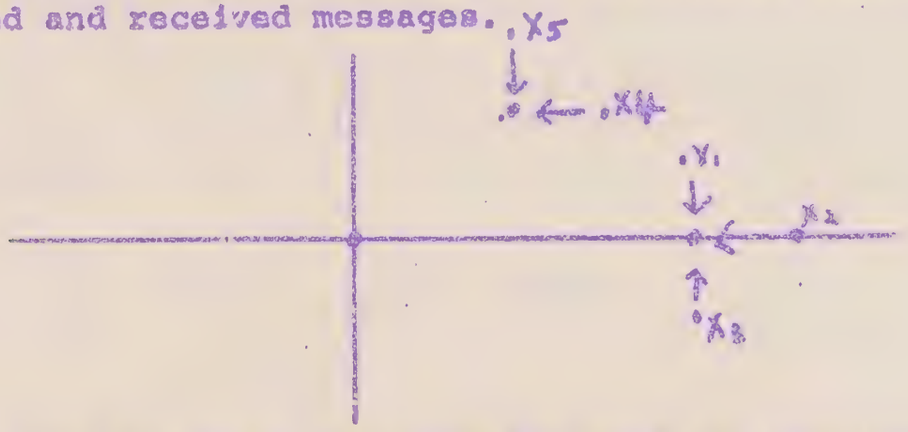
Where $W_1(t-1)$ and $W_2(t-1)$ have the same distribution as $X^R(t-1)$ but are dependent and W_3 is Poisson with mean λ and independent of W_1, W_2.

Finally;

$$X_d(t) = Z_1(t-1) + Z_2(t-1) + Z_3(t-1) + Z_4$$

Where $Z_1(t-1)$ and $Z_2(t-1)$ have the same distribution as $X^R(t-1)$, $Z_3(t-1)$ has the same distribution as $X_d^R(t-1)$ and depends on Z_1, Z_2 while Z_4 is Poisson with mean λ and is dependent of Z_1, Z_2, Z_3.

There are essentially two types of repeaters in this model; those on an axis and those off the axis. Since neighboring repeaters have dependent arrival distributions, we shall consider two types of "clusters" of repeaters and their associated random distributions of arrived and received messages.



We define the following multivariate density functions:

$$(6.3)\quad f_t(Y_1, Y_2, Y_3) = P[X_1(t)=Y_1,\ X_2(t)=Y_2,\ X_3(t)=Y_3]$$

(6.4) $f_t^R(z_1,z_2,z_3) = P[X_1^R(t)=z_1,\ X_2^R(t)=z_2,\ X_3^R(t)=z_3]$

(6.5) $g_t(w_1,w_2) = P[X_4(t)=w_1,\ X_5(t)=w_2]$

(6.6) $g_t^R(u_1,u_2) = P[X_4^R(t)=u_1,\ X_5^R(t)=u_2]$

where X_4, X_5 have the distributions of X as do X_1, X_2 while X_3 has the distribution of X_d. The initial conditions are:

(6.7) $$f_0(y_1,y_2,y_3) = \frac{e^{-\lambda}\, e^{-\lambda}\, e^{-\lambda}}{y_1!\, y_2!\, y_3!}, \qquad y_1, y_2, y_3 \geq 0$$

(6.8) $$g_0(w_1,w_2) = \frac{e^{-2\lambda}\, \lambda^{w_1+w_2}}{w_1!\, w_2!}, \qquad w_1, w_2, \geq 0.$$

Under type I slotting, we can develop the following relations between the f's and g's.

(6.9) $$f_t^R(z_1,z_2,z_3) = \sum_{y_1,y_2,y_3} P\left[(X_1^R(t)=z_1,\ X_2^R(t)=z_2,\ X_3^R(t)=z_3 \,\middle|\, X_1(t)=y_1,\ X_2(t)=y_2,\ X_3(t)=y_3\right]$$

$$P[X_1(t)=y_1,\ X_2(t)=y_2,\ X_3(t)=y_3]$$

$$= \sum_{y_1,y_2,y_3} \binom{y_1}{x_1}\binom{y_2}{x_2}\binom{y_3}{x_3} e^{-\frac{(x_1y_1+x_2y_2+x_3y_3)}{m}} \left(1-e^{-\frac{y_1}{m}}\right)^{y_1-x_1}$$

$$\left(1-e^{-\frac{y_2}{m}}\right)^{y_2-x_2} \left(1-e^{-\frac{y_3}{m}}\right)^{y_3-x_3} f_t(y_1,y_2,y_3)$$

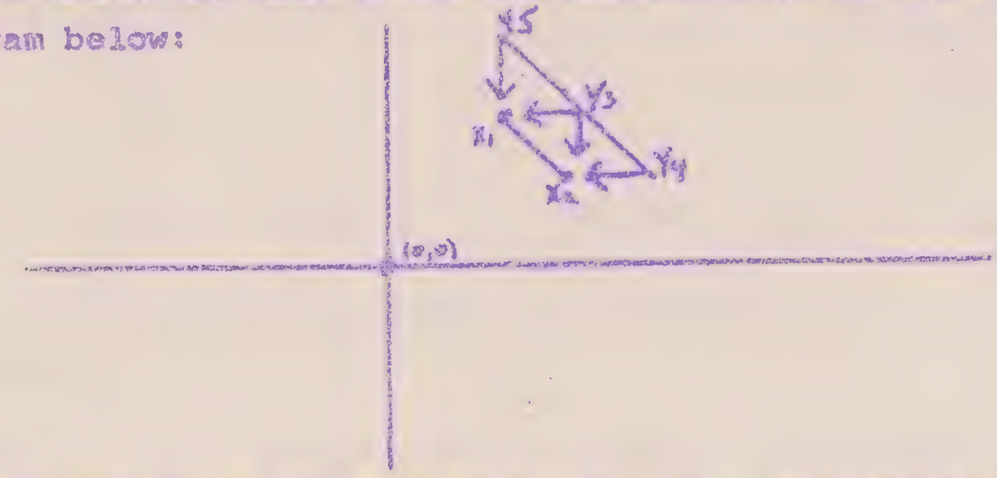
(6.10) $$g_t^R(u_1,u_2) = \sum_{w_1,w_2 \geq 0} P[X_4^R(t)=u_1,\ X_5^R(t)=u_2 \,|\, X_4(t)=w_1,\ X_5(t)=w_2]$$

$$\cdot\, P[X_4(t)=w_1,\ X_5(t)=w_2].$$

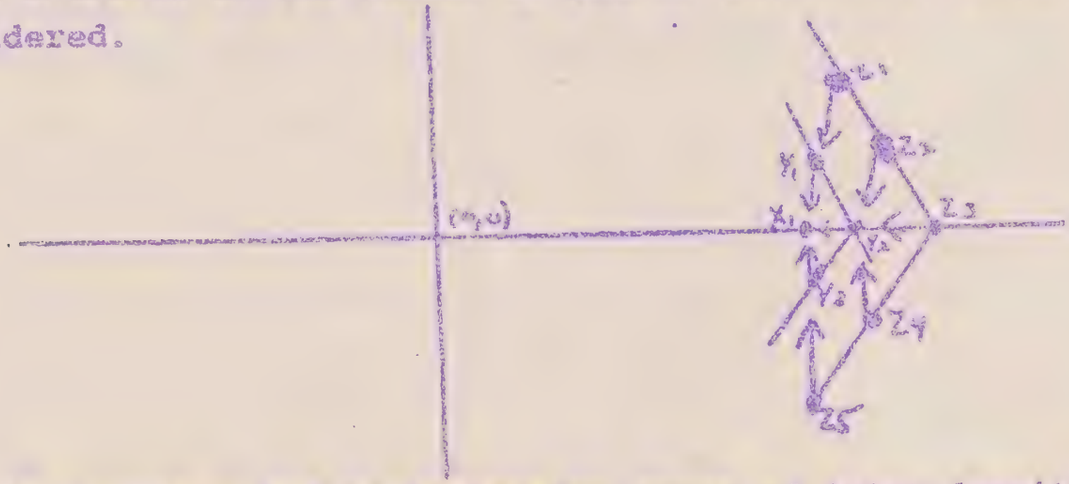
$$= \sum_{W_1, W_2 \geq 0} \binom{W_1}{U_1}\binom{W_2}{U_2} e^{\frac{-(W_1U_1+W_2U_2)}{m}} \left(1-e^{-\frac{W_1}{m}}\right)^{W_1-U_1}$$

$$\left(1-e^{-\frac{W_2}{m}}\right)^{W_2-U_2} \cdot g_t(W_1, W_2).$$

In order to determine the density functions (6.3, 6.4, 6.5, 6.6) we need equations relating these functions over time. In general, this is a difficult task since the distribution at say X_1, X_2 refer to diagram below:



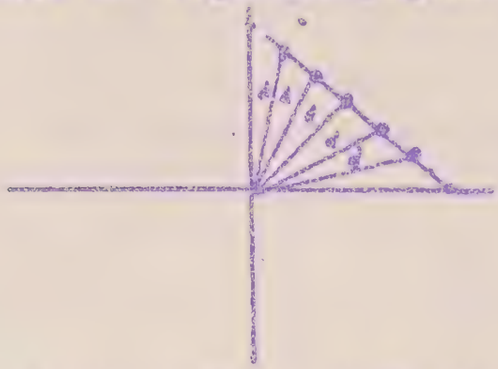
depends on the joint density of Y_3, Y_4, Y_5. Similarly, the density at points along the axis depend on distributions along a wedge which increase in the number of points (repeaters) which must be considered.



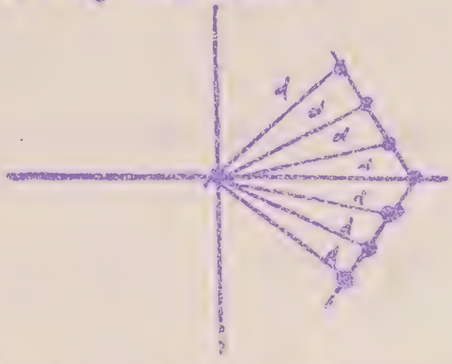
We are thus led to define two types of joint density functions of variable numbers of random variables.

We define the set of all points at the same distance from the origin as an isodesic set. Consecutive points on an isodesic line

are a sequence of points whose neighbors differ in distance by one unit from an axis. An isodesic wedge is a sequence of points on isodesic lines with a point on an axis and an equal number of points in neighboring quadrants separated by the axis.



isodesic line of 5 points



isodesic wedge of 7 points

For an isodesic line with k points at distance d define the joint density,

$$\bar{f}_t(X_1,\ldots,X_k) = P[\text{the number of messages arriving at each of the k points is } X_1, X_2,\ldots,X_k \text{ respectively}]$$

(Note that the order and distance are unimportant due to symmetries).

Denote the received versions by $\bar{f}^R_t(X_1,\ldots,X_k)$.

Similarly, for the wedge density define;

$\bar{g}_t(X_1,\ldots,X_k,Y,Z_1,\ldots,Z_k)$ be the joint density of arriving messages at time t at the points along the wedges and

$\bar{g}^R_t(X_1,\ldots,X_k,Y,Z_1,\ldots,Z_k)$ be the received joint densities. It is easy now to write difference equations over k and time (note that $k \le d-1$) otherwise f and g are not defined.

The initial conditions are independent over repeaters so that the joint densities are Poisson products.

$$(6.11)\quad \bar{f}_0(X_1,\ldots,X_k) = \frac{e^{-k\lambda}\,\lambda^{\sum_{i=1}^{k} X_i}}{X_1!\,X_2!\cdots X_k!} \qquad X_1,X_2,\ldots,X_k \ge 0,\ k=1,\ldots,d-1 .$$

$$(6.12)\quad \tilde{g}_0(x_1,\dots,x_k,y,z_1,\dots,z_k) = \frac{e^{-(2k+1)\lambda}\,\lambda^{\sum_{i=1}^{k} x_i+\sum_{i=1}^{k} z_i+y}}{x_1!\dots x_k!\; y!\; z_1!\dots z_k!}\,;$$

$$x_1,\dots,x_k,y,z_1,\dots,z_k \geq 0$$

$$k=0,\dots,d-1.$$

The received and arriving versions are connected by the following equations.

$$(6.13)\quad \tilde{f}_t^R(x_1,\dots,x_k) = \sum_{\substack{z_i\geq 0\\ i=1,2,\dots,k}} \binom{z_1}{x_1}\binom{z_2}{x_2}\cdots\binom{z_k}{x_k} e^{-(z_1x_1+z_2x_2+\dots+z_kx_k)}$$

$$\left(1-e^{-\frac{z_1}{m}}\right)^{z_1-x_1}\left(1-e^{-\frac{z_2}{m}}\right)^{z_2-x_2}\cdots\left(1-e^{-\frac{z_k}{m}}\right)^{z_k-x_k}$$

$$\tilde{z}_t(z_1,\dots,z_k)$$

$$(6.14)\quad \tilde{g}_t^R(x_1,\dots,x_k,y,z_1,\dots,z_k) = \sum_{\substack{y_i\geq 0\\ i=1,2,\dots k}} \sum_{\substack{w_j\geq 0\\ j=1,2,\dots,k}} \sum_{r\geq 0}$$

$$\binom{y_1}{x_1}\binom{y_1}{x_2}\cdots\binom{y_k}{x_k}\binom{r}{y}\binom{w_1}{z_1}\binom{w_2}{z_2}\cdots\binom{w_k}{z_k} e^{-\sum_{i=1}^{k}x_iy_i+y+\sum_{i=1}^{k}z_iw_i}$$

$$\left(1-e^{-\frac{y_1}{m}}\right)^{y_1-x_1}\left(1-e^{-\frac{y_2}{m}}\right)^{y_2-x_2}\cdots\left(1-e^{-\frac{y_k}{m}}\right)^{y_k-x_k}$$

$$\left(1-e^{-\frac{r}{m}}\right)^{r-y}\left(1-e^{-\frac{w_1}{m}}\right)^{w_1-z_1}\left(1-e^{-\frac{w_2}{m}}\right)^{w_2-z_2}\cdots\left(1-e^{-\frac{w_k}{m}}\right)^{w_k-z_k}$$

$$\tilde{g}_t(y_1,y_2,\dots,y_k,r,w_1,\dots,w_k).$$

The difference equations, over time are given by:

$$(6.15)\quad \hat{f}_t(X_1,\ldots,X_k) = \sum_{\substack{0\le W_i\le m \\ i=1,2,\ldots,k+1}} P[X_1,\ldots,X_k \mid W_1,\ldots,W_{k+1}]_{\text{received at } t-1} \cdot P[W_1,\ldots,W_k]_{\text{received at } t-1},$$

so that,

$$\hat{f}_t(X_1,\ldots,X_k) = \sum_{\substack{0\le W_i\le m \\ i=1,2,\ldots,k+1}} P[W_1+W_2+Y_1=X_1]\cdot P[W_2+W_3+Y_2=X_2]\cdots \cdot P[W_k+W_{k+1}+Y_k=X_k]\ \hat{f}^R_{t-1}(W_1,\ldots,W_{k+1})$$

$$(6.16)\quad = \sum_{\substack{0\le W_i\le m \\ i=1,2,\ldots,k+1 \\ W_i+W_{i+1}\le X_i}} \frac{e^{-\lambda k}\,\lambda^{\sum_{i=1}^{k} X_i}\,\lambda^{-W_1-W_{k+1}}\,\lambda^{-2\sum_{j=2}^{k} W_j}}{(X_1-W_1-W_2)!\,(X_2-W_2-W_3)!\cdots(X_k-W_k-W_{k+1})!} \cdot \hat{f}^R_t(W_1,\ldots,W_{k+1}) \qquad k=1,\ldots,d-1.$$

Similarly for the "wedge" density functions;

$$(6.17)\quad \hat{g}_t(X_1,\ldots,X_k,Y,Z_1,\ldots,Z_k) = \sum_{W,U,V} P[W_1+W_2+Y_1=X_1]\cdot P[W_2+W_3+Y_2=X_2]\cdots \cdot P[W_k+W_{k+1}+Y_k=X_k]\ P[U+X_{k+1}+Z_1+S=Y] \cdot P[V_1+V_2+U_1=Z_1]\ P[V_2+V_3+U_2=Z_2]\cdots \cdot P[V_k+V_{k+1}+U_k=Z_k] \cdot g^R_{t-1}(W_1,\ldots,W_{k+1},U,V_1,\ldots,V_{k+1}).$$

$$= \sum_{\substack{0 \le W_i \le m \\ W_i + W_{i+1} \le X_i \\ 1 \le i \le k+1}} \sum_{0 \le U \le m} \sum_{\substack{0 \le V_i \le m \\ V_i + V_{i+1} \le Z_i \\ 1 \le i \le k+1}} \lambda^{-(2k+1)\lambda^{\sum_{i=1}^{k} X_i}\lambda^{\sum_{i=1}^{k} Z_i}\lambda^{-W_1 - W_{k+1}}}$$

$$\lambda^{-V_1 - V_{k+1}} \lambda^{-2\sum_{j=2}^{k} W_j} \lambda^{-2\sum_{j=2}^{k} V_j} \lambda^{Y-U-X_{k+1}-Z_1} \cdot$$

$$\cdot g_{t-1}^{\ge R}(W_1, \ldots, W_{k+1}, U, V_1, \ldots, V_{k+1})$$

all divided by,

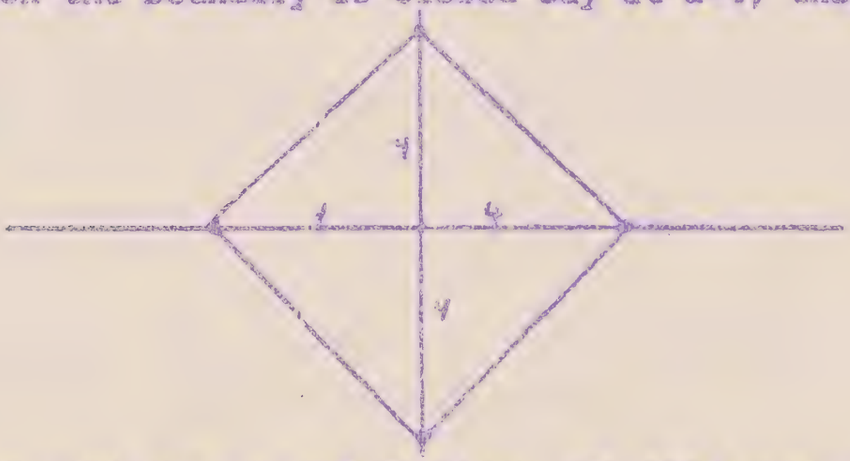
$$(X_1 - W_1 - W_2)!\ (X_2 - W_2 - W_j)! \ldots (X_k - W_k - W_{k+1})!\ (Z_1 - V_1 - V_2)! \ldots (Z_k - V_k - V_{k+1})!$$

$$(Y - U - X_{k+1} - Z_1)!.$$

Theoretically, these equations can be solved since all initial condition (t=0) have been given and recurrences in time are derived.

VII. REMARKS ON THE CLOSED BOUNDARY PROBLEM:

When the boundary is closed say at d=4, there are 16+12+8+4=40



repeaters and one ground station. In this case of course, the recurrence relations of section VI do not hold due to the loss of symmetry. The equations however, can be modified using the same techniques of conditioning as in section VI, the results

for P_{kj} and $\bar{P}_{kj}$ are still valid and provide the link between arrivals and receptions.

The inputs at the boundary are all Poisson and independent, thus all wedge and line joint densities are mere products. Since all initial distributions are still valid as Poisson products for lines and wedges, the appropriate difference equations can be repetitively solved. It is expected that steady state behaviour would be obtained rather quickly and simulation type methods could be effectively used to solve the very complicated set of time dependent difference equations.

VIII. SOME USEFUL INEQUALITIES:

Let $X_0(t)$ be the number of messages which arrive at the origin at time t and $X_d(t)$ be the number of messages arriving at a repeater on the axis at distance d at time t. Using type I slotting, we can derive an inequality for the number of messages received at distance d. Denote by a superscript R the received versions of $X_0(t)$ and $X_d(t)$. We have,

$$E[X_0(t)] = 4E[X_1^R(t-1)] + \lambda \ .$$

However,

$$E[X_d^R(t)] = \sum_{k=0}^{\infty} E[X_d^R(t) \mid X_d(t)=k] \ P[X_d(t)=k].$$

Since,

$$E[X_d^R(t) \mid X_d(t)=k] = \sum_{j=0}^{k} j \ P_{kj} = ke^{-\frac{k}{m}}$$

We obtain,

$$E[X_d^R(t)] = \sum_{k=0}^{\infty} k \, e^{-\frac{k}{m}} \ P[X_d(t)=k]$$

$$= E[X_d(t) \cdot e^{-\frac{X_d(t)}{m}}]$$

Which is a derivative of the Laplace transform of the density function of $X_d(t)$.

A trivival inequality now follows from using the bound $e^{-X} \leq 1-X+\frac{X^2}{2}$ and hence,

$$E[X_t^R(t)] = E[X_d(t)\ (1-\frac{X_d(t)}{m}+\frac{X_d^2(t)}{2m^2}\ )]$$

$$= E[X_d(t)] - E[\frac{X_d^2(t)}{m}] + E[\frac{X_d^3(t)}{2m^2}].$$

Perhaps, this will be useful when the third moments of the simplified wedge densities can be obtained.
Since,

$$X_d(t) = Y_1^R(t-1) + Y_2^R(t-1) + Y_3^R(t-1) + Y$$

(Where Y_1^R, Y_2^R are identically distributed with distribution of $X^R(t)$ and $Y_3(t-1)$ is distributed as $X_d^R(t)$, and Y is Poisson and independent of Y_1^R, Y_2^R, Y_3^R), the third moment of $X_d(t)$ involves all kinds of cross correlations which seem difficult to simplify.

IX. SUMMARY AND CONCLUSIONS:

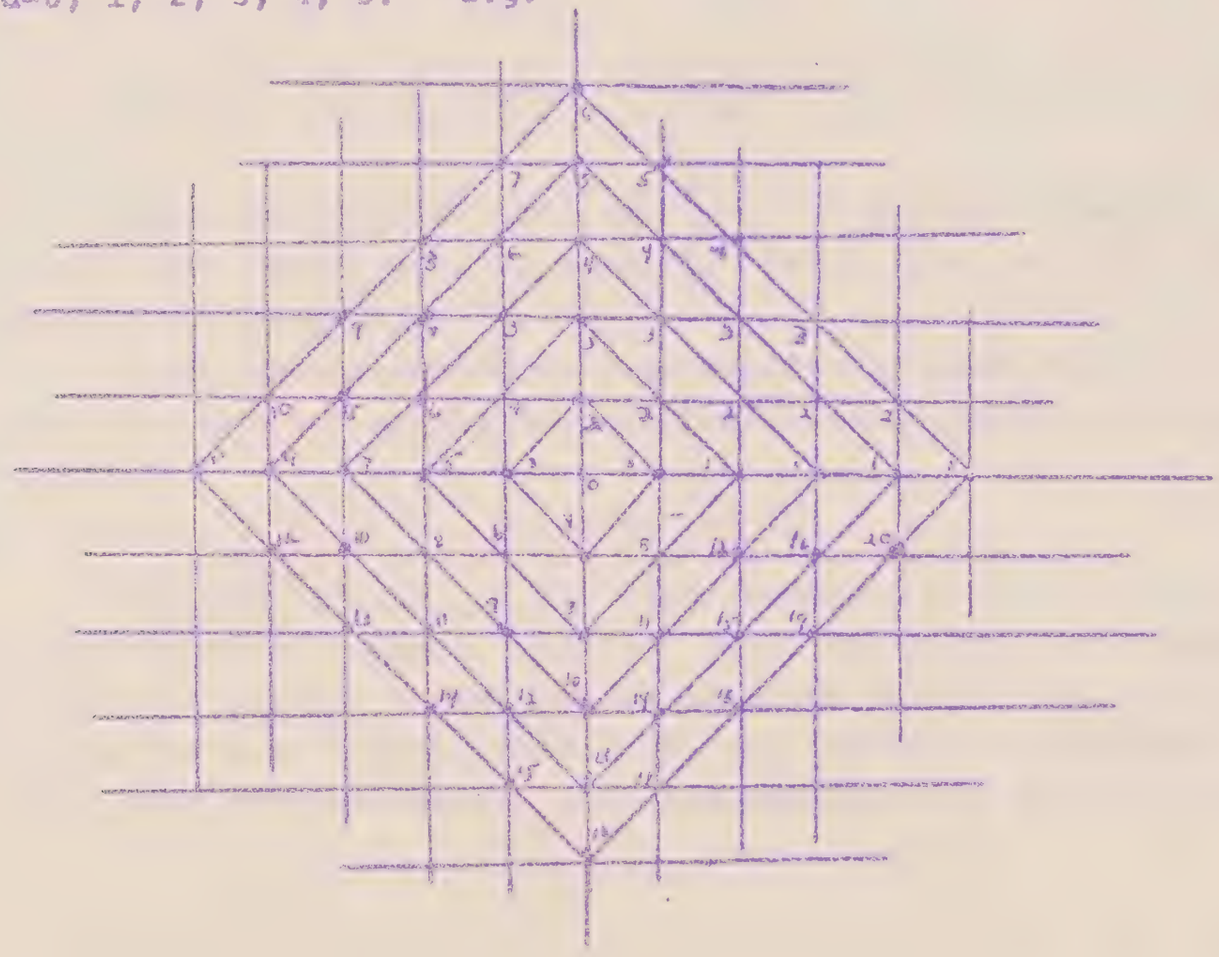
On a theoretical basis we have solved all problems posed in section II. Unfortunately, all of the formulae are in non-tractable form. Even slight simplifications seem quite difficult to obtain. The transfer probabilities which relate arriving messages to received messages are Stirling types of numbers and no simplification for the sums are available. For purposes of computation, recurrence formulae are available as well as the sum formulae given in sections III and IV.

For reasonably small boundaries, say 40 repeaters it seems that computer analyses should be reasonable to attempt and that steady state solutions should appear rather quickly. Much theoretical effort can still yield a deeper understanding of the model discussed.

IX. COMPUTER ANALYSIS: A CLOSED BOUNDARY SITUATION:

The results and complexities observed in sections II to VII indicate the usefulness of computer analysis. An interesting and perhaps symbolic special case for computer analysis might be obtained for closing the boundary at d=5 steps from the origin. The distribution for each of the two modes (section III and IV) can be computed and used as transfer functions from "arrived" to "received" messages. Of course, these transfers will take place by random sampling from those distributed and hence one less type of simulation will be required.

The computer analysis will begin once we have numbered the 61 repeaters which lie at a distance of five or less units from the origin. We do this in a counter clockwise direction beginning with d=0, d=1, d=2, d=3, d=4, d=5 and j coordinates 1, 2,...,4d d=0, 1, 2, 3, 4, 5. e.g.



The repeaters at d=5 do not have messages arriving from other stations. They only receive their own traffic at Poisson rate. Repeaters at $d < 5$ which are on the "axes" (denoted by circles) have messages arriving from the three neighbors at d+1, as well as their own Poisson traffic.

Repeaters off the axes at distance $d < 5$ have input at each time point from the two neighbors at d+1 as well as their own Poisson traffic.

The network is activated at t=0 by having random Poisson arrivals with mean λ at each of the 61 repeaters. This input traffic at each repeater is converted to received messages in each of the two possible modes for different values of m by use of the "transfer functions".

$$(1)\quad P_{kj} = \frac{\binom{m}{j}}{m^k} \sum_{\nu=0}^{\min(k-j,\,m-j)} (-1)^{\nu} \cdot \binom{m-j}{\nu} \frac{k!}{(k-j-\nu)!} (m-j-\nu)^{k-j-\nu} \quad ;$$

$$(2)\quad P^{*}_{kj} = \binom{m}{j} \sum_{\nu=0}^{j} (-1)^{\nu} \binom{j}{\nu} \left(\frac{j-\nu}{m}\right)^{k} , \qquad j=0,\ 1,\ 2,\ldots,\min(k,m)$$

These calculations give us $P^{R}_{(d,j)}(0)$ for all repeaters with coordinates (d,j), d=1, 2, 3, 4, 5, j=1,...,4d, an (0,0) the station at the origin.

We can now determine message traffic at each repeater by using equations which describe message transmission in direction of the "origin".

For Time t=1

When d=5; j=1, 2,...,20 , the repeaters at d=5 receive only their generated Poisson traffic. Thus, for time 1 we generate 61 Poisson traffic numbers which describe direct (i.e., at the source) message input. When $d \leq 4$, the repeater at coordinates (d,j) also receive traffic from its neighbors at further distance

by one unit. The following equations describe messages arriving at each repeater for arbitrary time t 1.

On the Axis:

$$P_{(0,0)}(t) = P^{R}_{(1,1)}(t-1)+P^{R}_{(1,2)}(t-1)+P^{R}_{(1,3)}(t-1)+P^{R}_{(1,4)}(t-1)+\text{Poisson}$$

At d=1 ($P_{1,1}$, $P_{1,2}$, $P_{1,3}$, $P_{1,4}$)

$$P_{(1,1)}(t) = P^{R}_{(2,1)}(t-1)+P^{R}_{(2,2)}(t-1)+P^{R}_{(2,8)}(t-1)+\text{Poisson}$$

$$P_{(1,2)}(t) = P^{R}_{(2,2)}(t-1)+P^{R}_{(2,3)}(t-1)+P^{R}_{(2,4)}(t-1)+\text{Poisson}$$

$$P_{(1,3)}(t) = P^{R}_{(2,4)}(t-1)+P^{R}_{(2,5)}(t-1)+P^{R}_{(2,6)}(t-1)+\text{Poisson}$$

$$P_{(1,4)}(t) = P^{R}_{(2,6)}(t-1)+P^{R}_{(2,7)}(t-1)+P^{R}_{(2,8)}(t-1)+\text{Poisson}$$

At d=2 $P_{2,1}$, $P_{2,3}$, $P_{2,5}$, $P_{2,7}$

$$P_{(2,1)}(t) = P^{R}_{(3,1)}(t-1)+P^{R}_{(3,2)}(t-1)+P^{R}_{(3,12)}(t-1)+\text{Poisson}$$

$$P_{(2,3)}(t) = P_{(3,3)}(t-1)+P^{R}_{(3,4)}(t-1)+P^{R}_{(3,5)}(t-1)+\text{Poisson}$$

$$P_{(2,5)}(t) = P^{R}_{(3,6)}(t-1)+P^{R}_{(3,7)}(t-1+P^{R}_{(3,8)}(t-1)+\text{Poisson}$$

$$P_{(2,7)}(t) = P^{R}_{(3,9)}(t-1)+P^{R}_{(3,10)}(t-1)+P^{R}_{(3,11)}(t-1)+\text{Poisson}$$

At d=3 $P_{3,1}$, $P_{3,4}$, $P_{3,7}$, $P_{3,10}$

$$P_{(3,1)}(t) = P^{R}_{(4,1)}(t-1)-P^{R}_{(4,2)}(t-1)+P^{R}_{(4,16)}(t-1)+\text{Poisson}$$

$$P_{(3,4)}(t) = P^{R}_{(4,4)}(t-1)+P^{R}_{(4,5)}(t-1)+P^{R}_{(4,6)}(t-1)+\text{Poisson}$$

$$P_{(3,7)}(t) = P^{R}_{(4,8)}(t-1)+P^{R}_{(4,9)}(t-1)+P^{R}_{(4,10)}(t-1)+\text{Poisson}$$

$$P_{(3,10)}(t) = P^{R}_{(4,12)}(t-1)+P^{R}_{(4,13)}(t-1)+P^{R}_{(4,14)}(t-1)+\text{Poisson}$$

At d=4 $P_{(4,1)}(t)$, $P_{(4,5)}(t)$, $P_{(4,9)}(t)$, $P_{(4,13)}(t)$;

$$P_{(4,1)}(t) = P^{R}_{(5,1)}(t-1)+P^{R}_{(5,2)}(t-1)+P^{R}_{(5,20)}(t-1)+\text{Poisson}$$

$$P_{(4,5)}(t) = P^{R}_{(5,6)}(t-1)+P^{R}_{(5,5)}(t-1)+P^{R}_{(5,7)}(t-1)+\text{Poisson}$$

$$P_{(4,9)}(t) = P^{R}_{(5,10)}(t-1)+P^{R}_{(5,11)}(t-1)+P^{R}_{(5,12)}(t-1)+\text{Poisson}$$

$$P_{(4,13)}(t) = P^{R}_{(5,15)}(t-1)+P^{R}_{(5,16)}(t-1)+P^{R}_{(5,17)}(t-1)+\text{Poisson}$$

Off the Axes:

$$P_{(d,j)}(t) = P^{R}_{(d+1,j)}(t-1+P^{R}_{(d+1,j+1)}(t-1)+\text{Poisson}; \quad j=2,3,\ldots,d; \quad d=2,3,4.$$

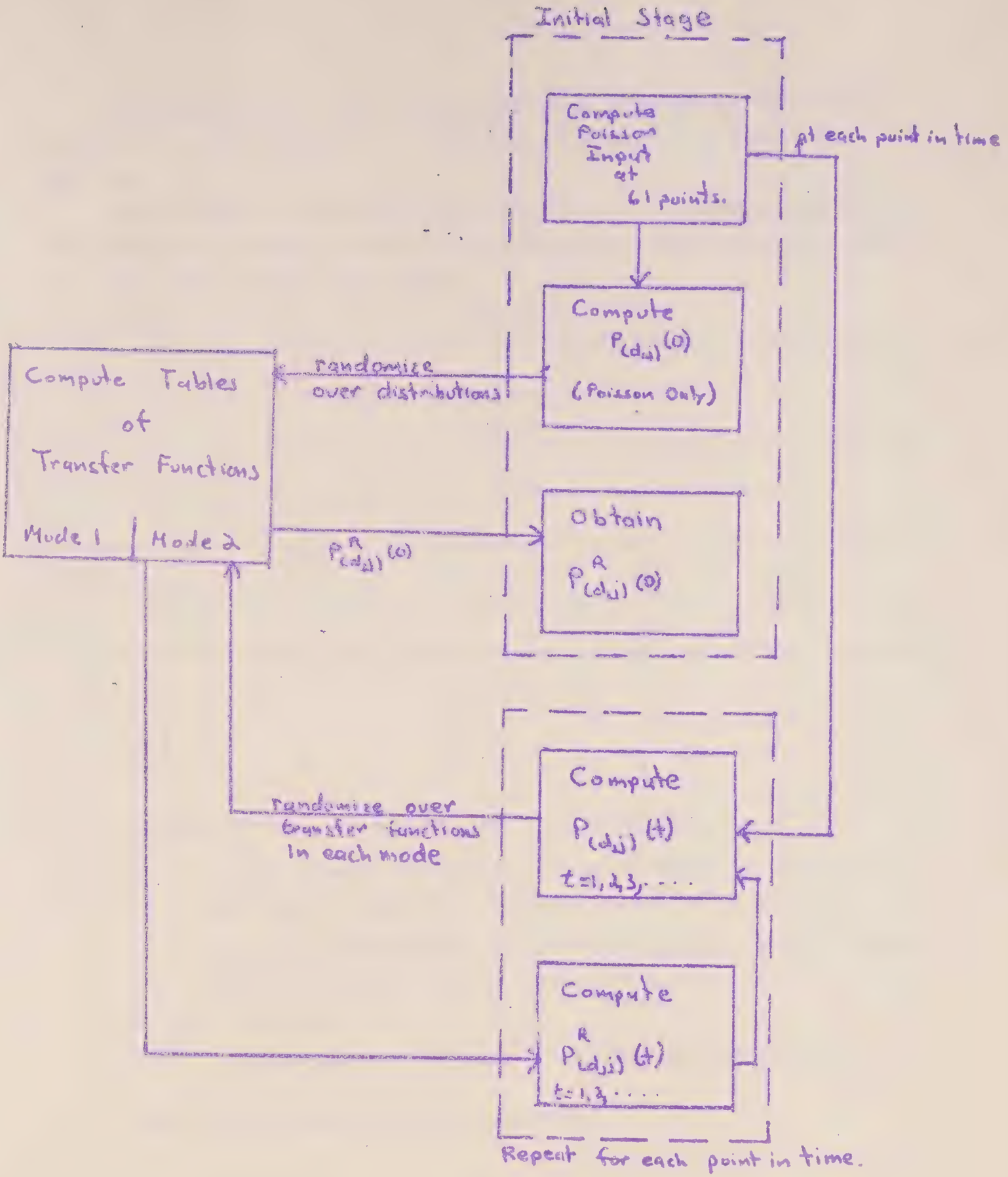
$$P_{(d,j)}(t) = P^{R}_{(d+1,j+1)}(t-1)+P^{R}_{(d+1,j+2)}(t-1)+\text{Poisson}; \quad j=d+2,d+3,..2d; \quad d=2,3,4.$$

$$P_{(d,j)}(t) = P^{R}_{(d+1,j+2)}(t-1)+P^{R}_{(d+1,j+3)}(t-1)+\text{Poisson}; \quad j=2d+2,..3d; \quad d=2,3,4.$$

$$P_{(d,j)}(t) = P^{R}_{(d+1,j+3)}(t-1)+P^{R}_{(d+1,j+4)}(t-1)+\text{Poisson}; \quad j=3d+2,..4d; \quad d=2,3,4.$$

These equations relate arriving and received messages over neighboring time points and repeater. Thus, the arriving number of messages can be computed in the grid at each point in time and each repeater.

In terms of a flow diagram, the procedure for analyzing this and all finite grids follows:


Initial Stage
Compute Poisson Input at 61 points.
at each point in time
Compute $P_{(d_{ij})}(0)$ (Poisson Only)
Compute Tables of Transfer Functions
Mode 1
Mode 2
randomize over distributions
$P^R_{(d_{ij})}(0)$
Obtain $P^R_{(d_{ij})}(0)$
randomize over transfer functions in each mode
Compute $P_{(d_{ij})}(t)$ $t=1,2,3,\ldots$
Compute $P^R_{(d_{ij})}(t)$ $t=1,2,\ldots$
Repeat for each point in time.

The parameters are, λ = mean Poisson arrival at each time point, at each repeater, m the number of slots in each mode one and two.

The output of the computer analysis is processed and presented in two forms, tabular and graphical. The tabular format is" for each m and λ and mode,

t	0	1	2	3	4	5	6	7	. . . .
$P_{(0,0)}(t)$									. . .
$P^{R}_{(0,0)}(t)$									
$P_{(1,1)}(t)$									
$P^{R}_{(1,1)}(t)$									
$P_{(2,2)}(t)$									
$P^{R}_{(2,2)}(t)$									
. . .									

Various graphic analyses are also obtained.

A. A graph of arrived and received messages at the origin as a function of time for various values of m and λ.

B. A frequency histogram of arrivals off the axis. There are 24 points of the axis at distance 2, 3, 4,.
We take for each time t;

$$f(x) = \frac{\text{number of stations with x arrivals at time t}}{24}$$

This is plotted for each time point.

C. The same histograms as in b except on the axis. There are 16 points on the axes at distances 1, 2, 3, 4.

D. The mean number of arrived and received messages $\hat{\lambda}(t)$ and $\hat{\lambda}^{R}(t)$ as a function of time on and off the axis. These are given by,

$$\hat{\lambda}(t) = \sum_{x=1} xf(x), \quad \hat{\lambda}^{R}(t) = \sum_{y=1} xf^{R}(x)$$

where $f(x)$ if the frequency of arrivals and $f^{R}(x)$ is the frequency of received messages.

$$\hat{\lambda}_{A}(t) = \sum_{x=1} xf_{A}(x), \quad \hat{\lambda}_{A}{}^{R}(t) = \sum_{x=1} xf_{A}{}^{R}(x),$$

where $f_{A}(x)$ and $f_{A}{}^{R}(x)$ are frequencies on the axis of arriving and received messages. Some numerical results follow.

REFERENCES

1. "Combinatorial Aspects of Message Flow in Packet Radio Nets" Part I: Network Analysis Corporation 24 Jan., 1973, ARPA Network Information Center, Packet Radio Temporary Note 6, NIC #13631.

2. Feller, W.; "An Introduction to Probability Theory and Its Applications", 3rd Ed., Vol. I; Wiley and Sons, N. Y.

APPENDIX: SUMMARY OF INITIAL COMPUTER ANALYSIS:

Attached, are two curves which represent a summary of data compiled from a preliminary computer investigation of a closed grid network. The grid selected for initial analysis is the closed boundary grid at distance five. We combined computer runs with the closed form theoretical analyses of sections three and four of this report to obtain some observations of network behaviour.

The first six curves represent a study of messages arriving and being received at the origin (fixed ground station) as a function of time. We used 20 computer runs for each of the first fifty time units. In this initial study the number of slots was kept fixed at 100, but λ (the mean number of messages originating at a given repeater) was set at 10, 20 and 30. All calculations were carried out for mode 1 and mode 2.

The message flow and reception at the origin settle down at about t=4 and remained relatively constant. For λ=10 the number of arriving messages seemed to have a mean at about 155 and the number of received messages averaged to about 31. Since the system behaviour for λ=10, m=100 settled down so quickly it seems reasonable to combine all time point data past t=10 to estimate the probability density function of arrivals and receptions at the origin in each of modes 1 and 2 when λ=10. The curves would seem to indicate asymptotic Poisson behaviour with means about 31, 155 in mode 1 and about 100, 300, in mode 2 respectively. Saturation occurs quickly in mode 2 for λ=10 or more. These results are summarized in the last four curves of probability density functions.

AREAS OF FUTURE ANALYSIS; THEORETICAL AND COMPUTER:

The results presented in the computer analysis are only a beginning to the type of results that can be obtained using the same basic program. There is the obvious broad general area of

studying message flow at the origin as a function of the parameters λ, m and mode for the present model and its extensions. More specifically, we have not yet studied the probability of message survival as a function of distance of origination, time of origination, λ, m and mode. This area is prime for our next analysis.

Another area yet unexplored with the present model is the effect of various types of breakdowns of parts of the network. This can be studied easily since all nodes are labeled and hence can easily be removed from the network calculations for an arbitrary desired length of time. The effect of such breakdowns can be analyzed in terms of all factors mentioned above as well as recovery time of the system.

The model can also be generalized and extended. Two specific directions are of interest.

A. Message Flow from the Fixed Ground Station To The Repeater Network:

We can repeat an analysis of the network when messages are flowing out from the origin. Questions similar to those posed for the inward model can be posed and analyzed.

B. Message Flow In Both Directions:

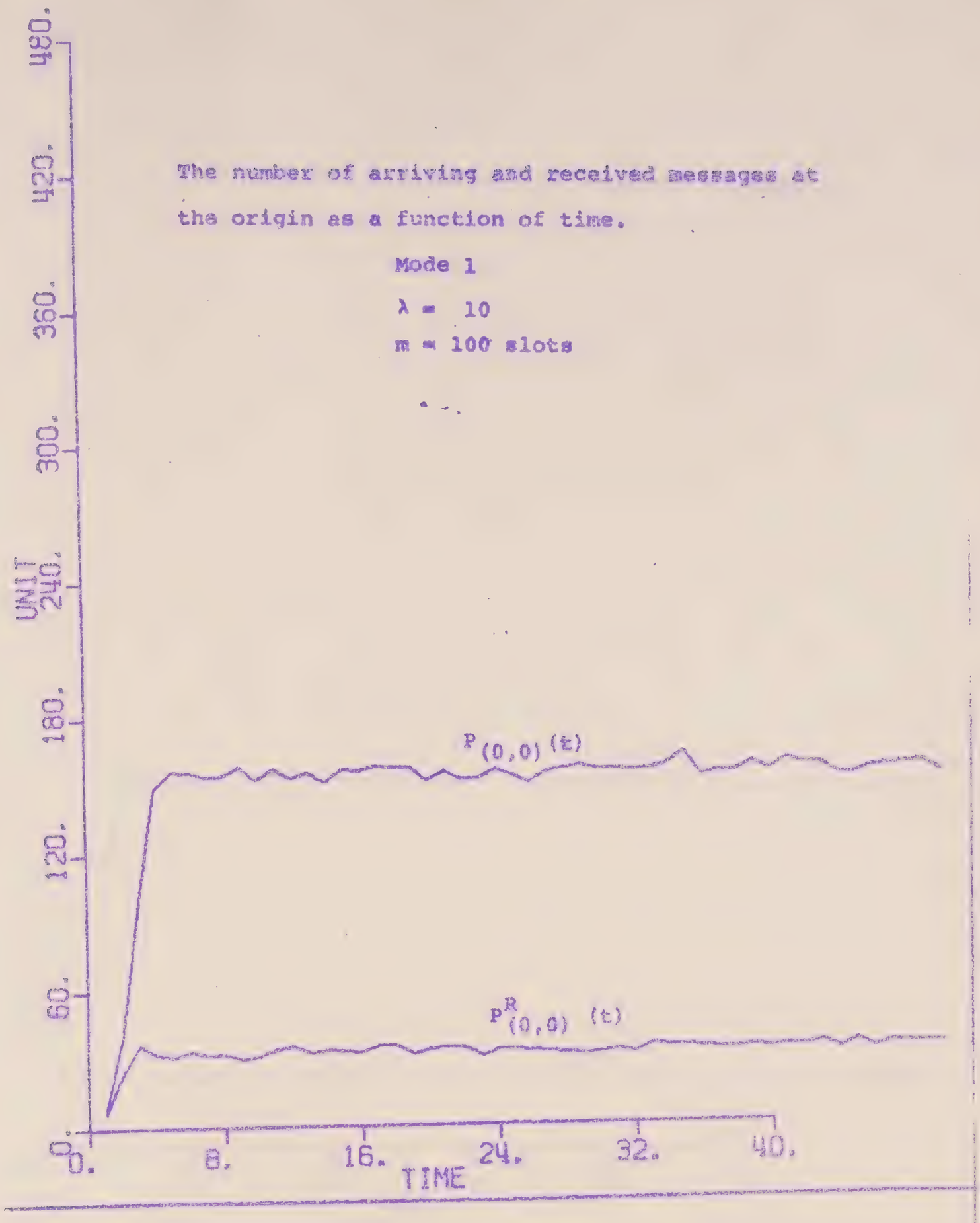
We can study the same model when messages are passing through repeaters to and from the origin. This network can be studied under various operating conditions which include one and two frequencies.

The number of arriving and received messages at the origin as a function of time.

Mode 1

λ = 10

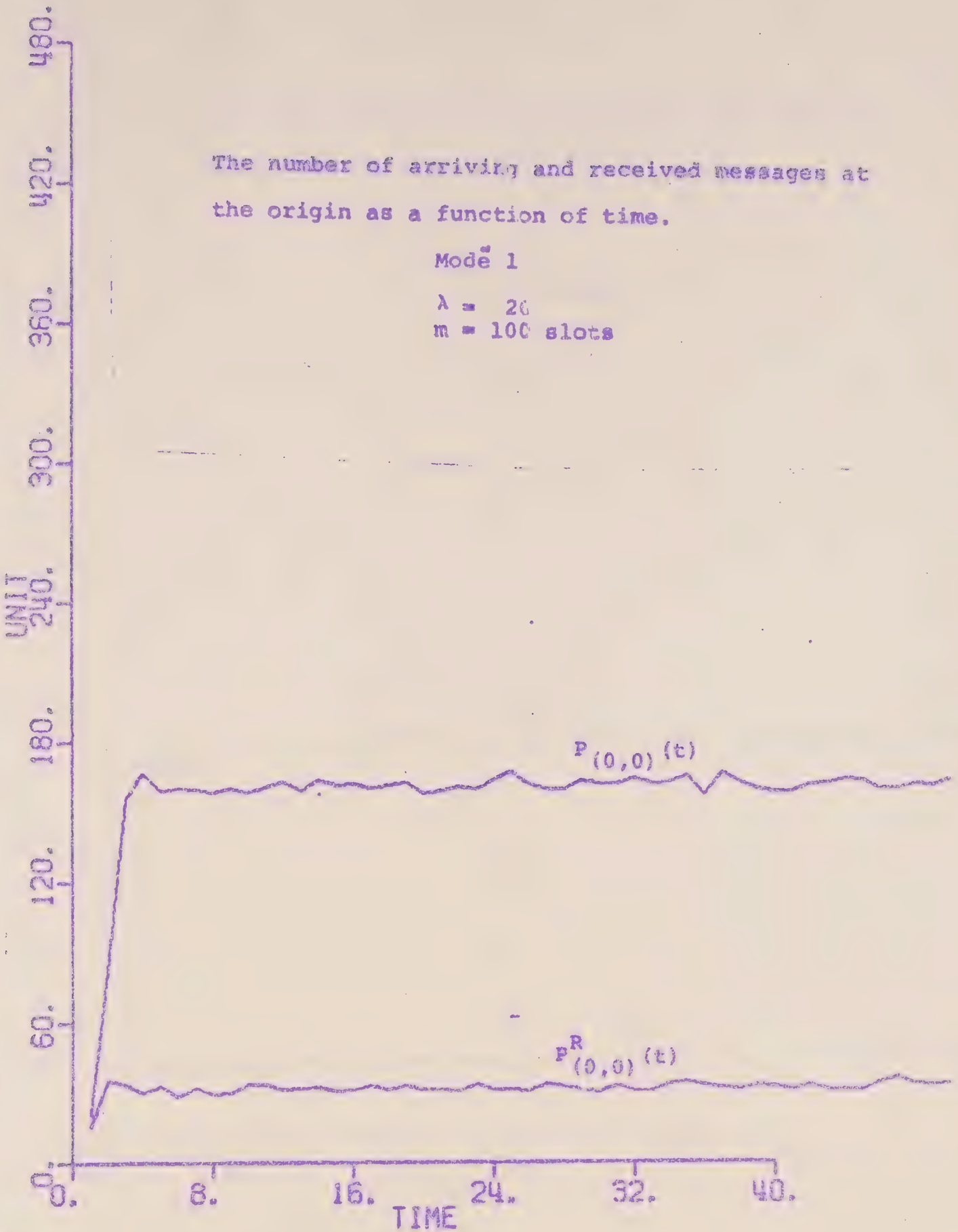
m = 100 slots



The number of arriving and received messages at the origin as a function of time.

Mode 1

λ = 20
m = 100 slots

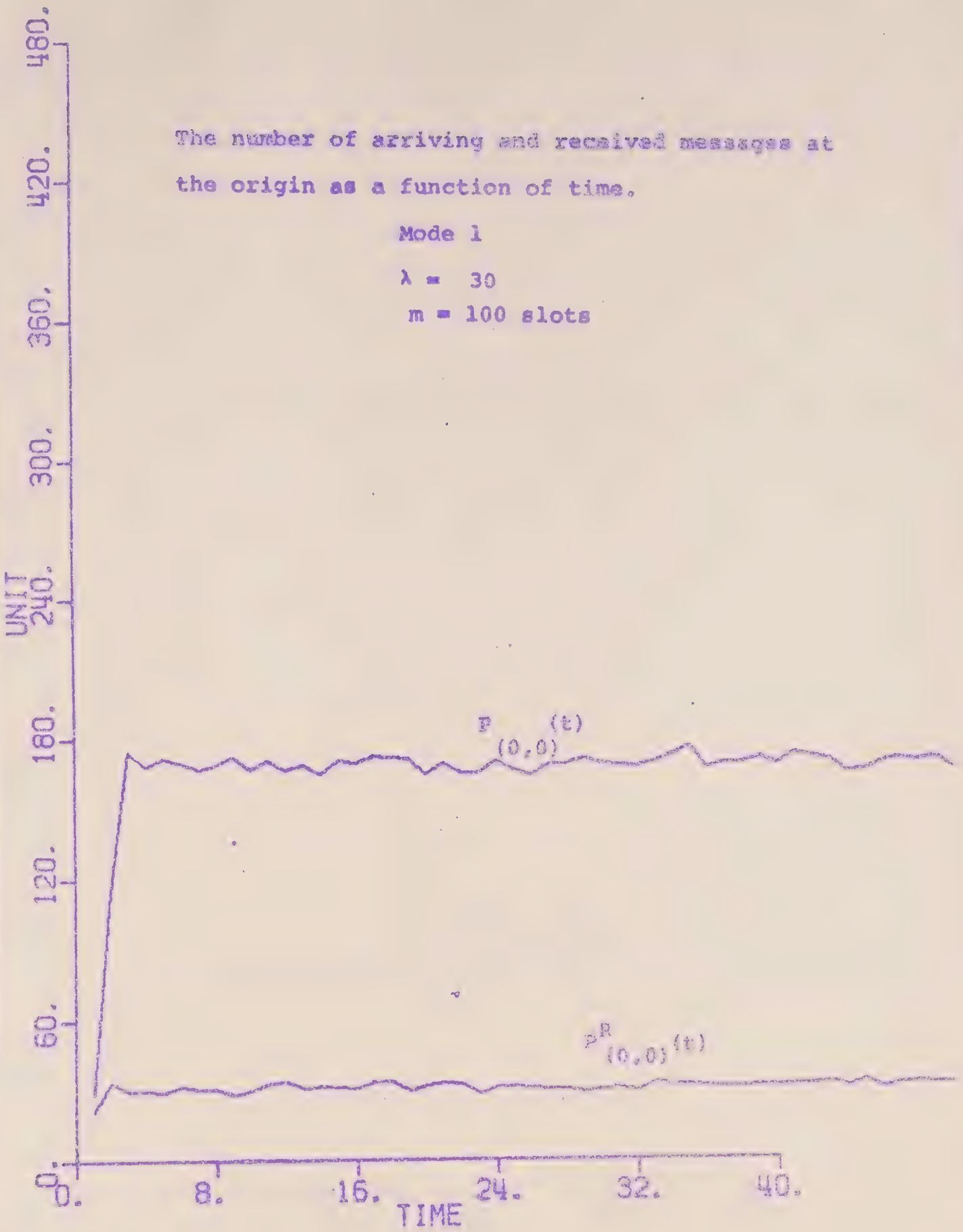


The number of arriving and received messages at the origin as a function of time.

Mode 1

λ = 30

m = 100 slots

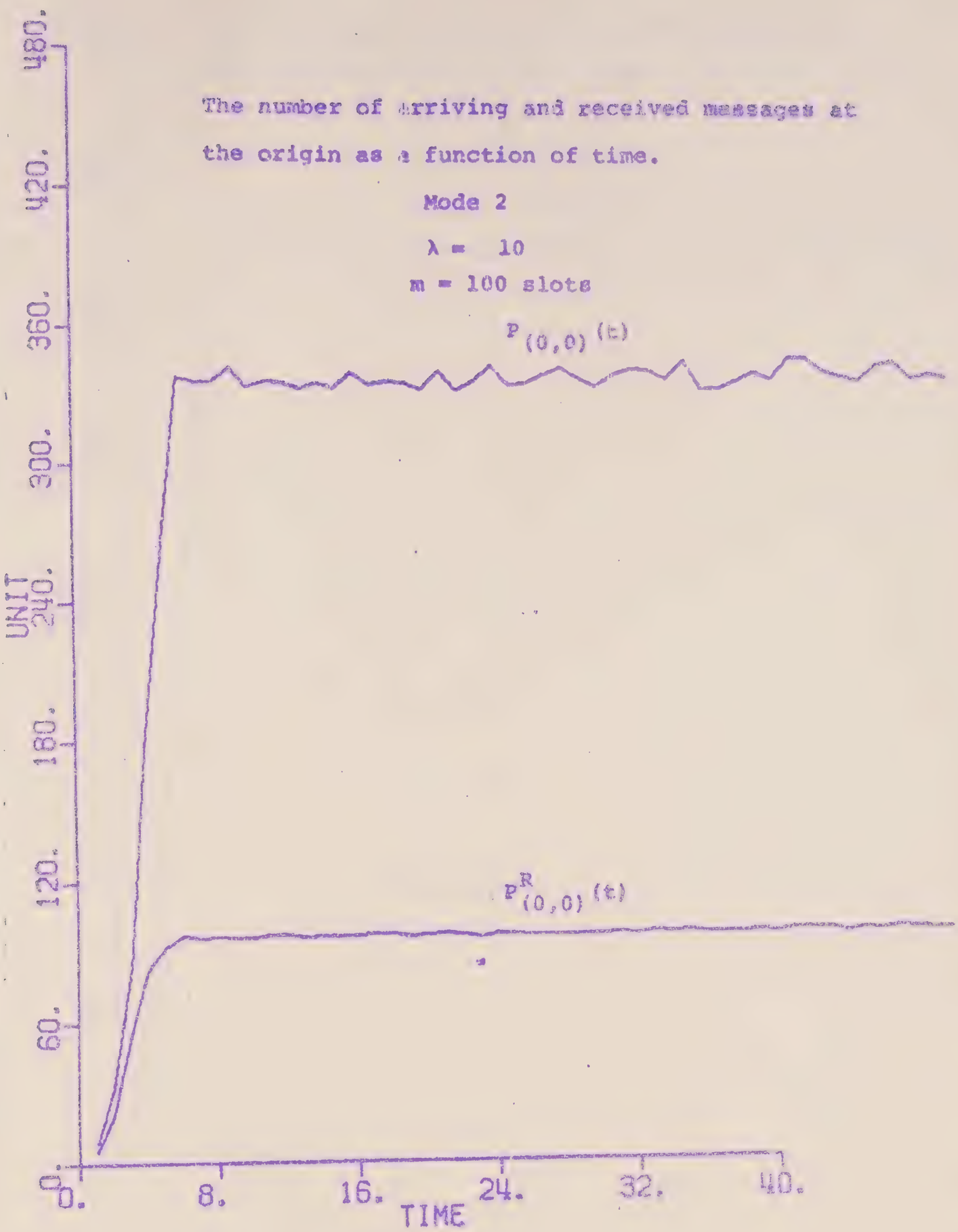


The number of arriving and received messages at the origin as a function of time.

Mode 2

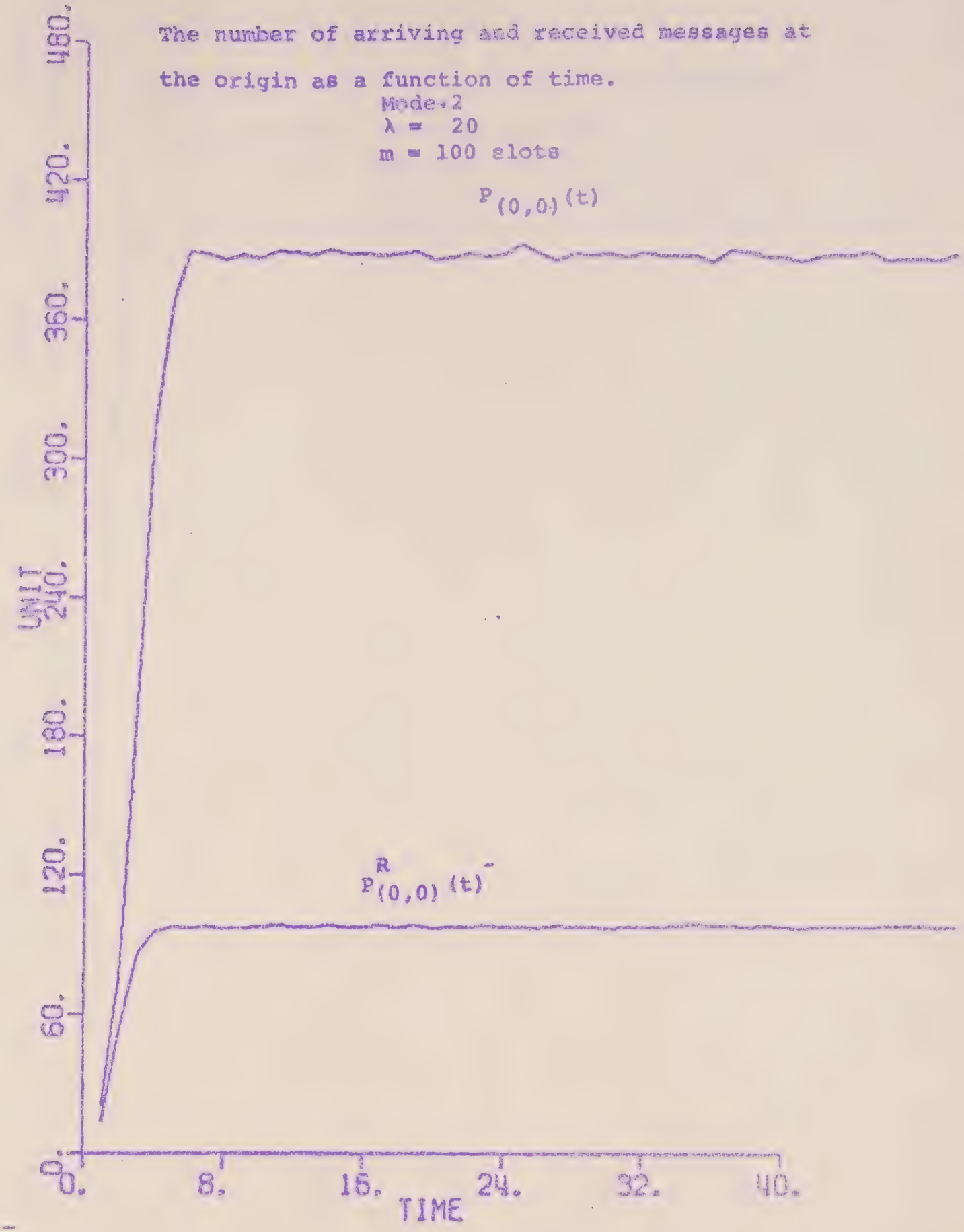
λ = 10

m = 100 slots

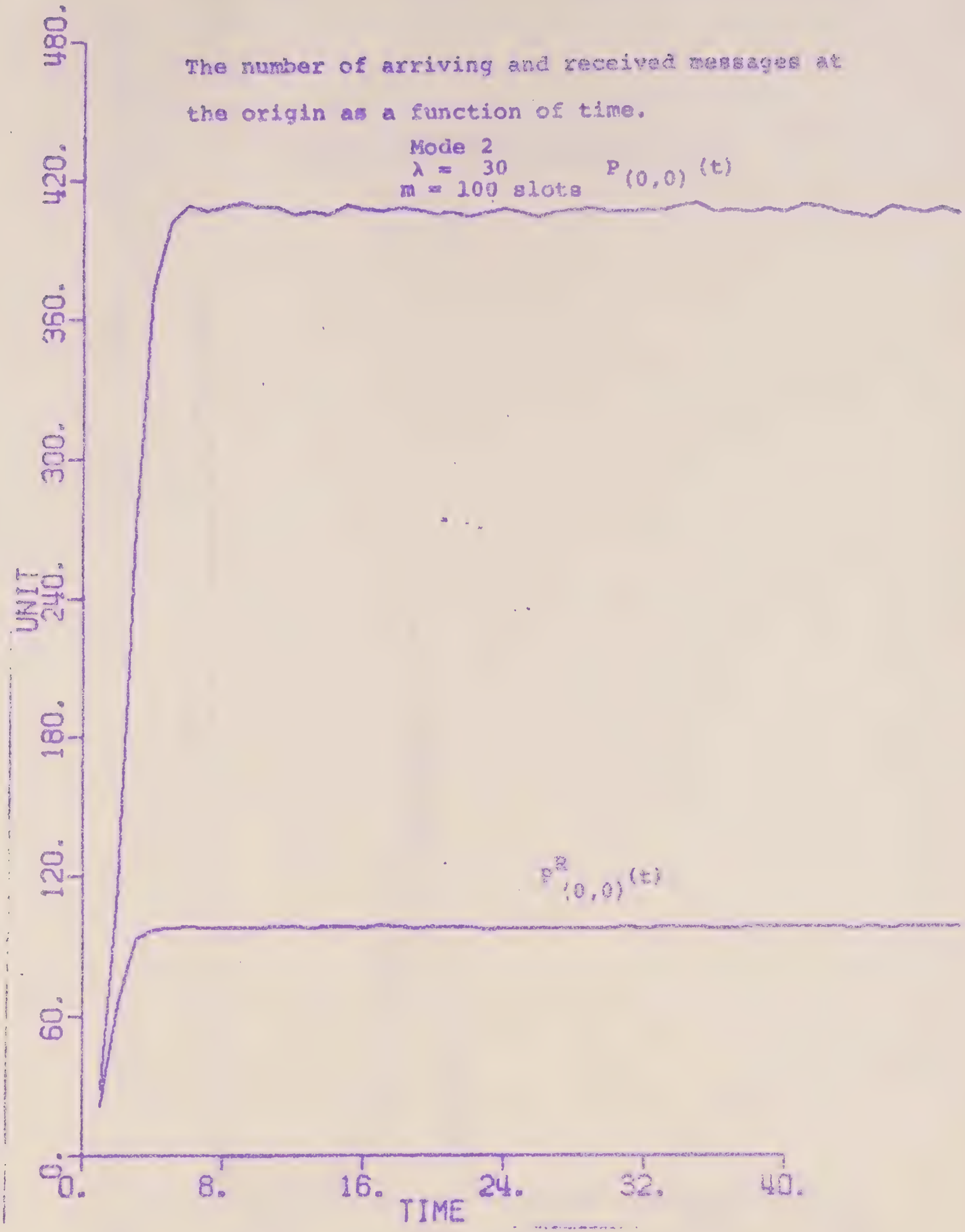


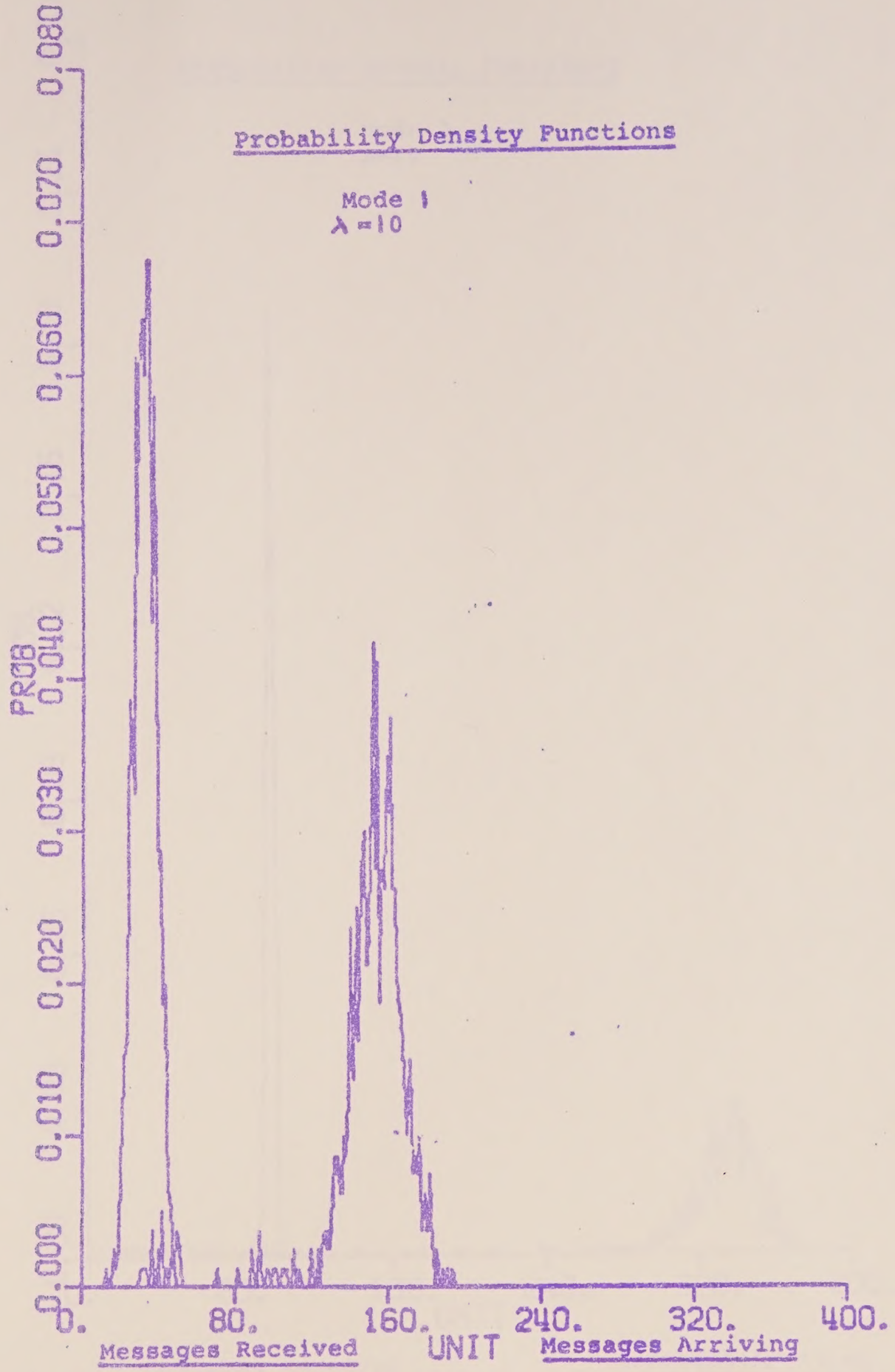
The number of arriving and received messages at the origin as a function of time.

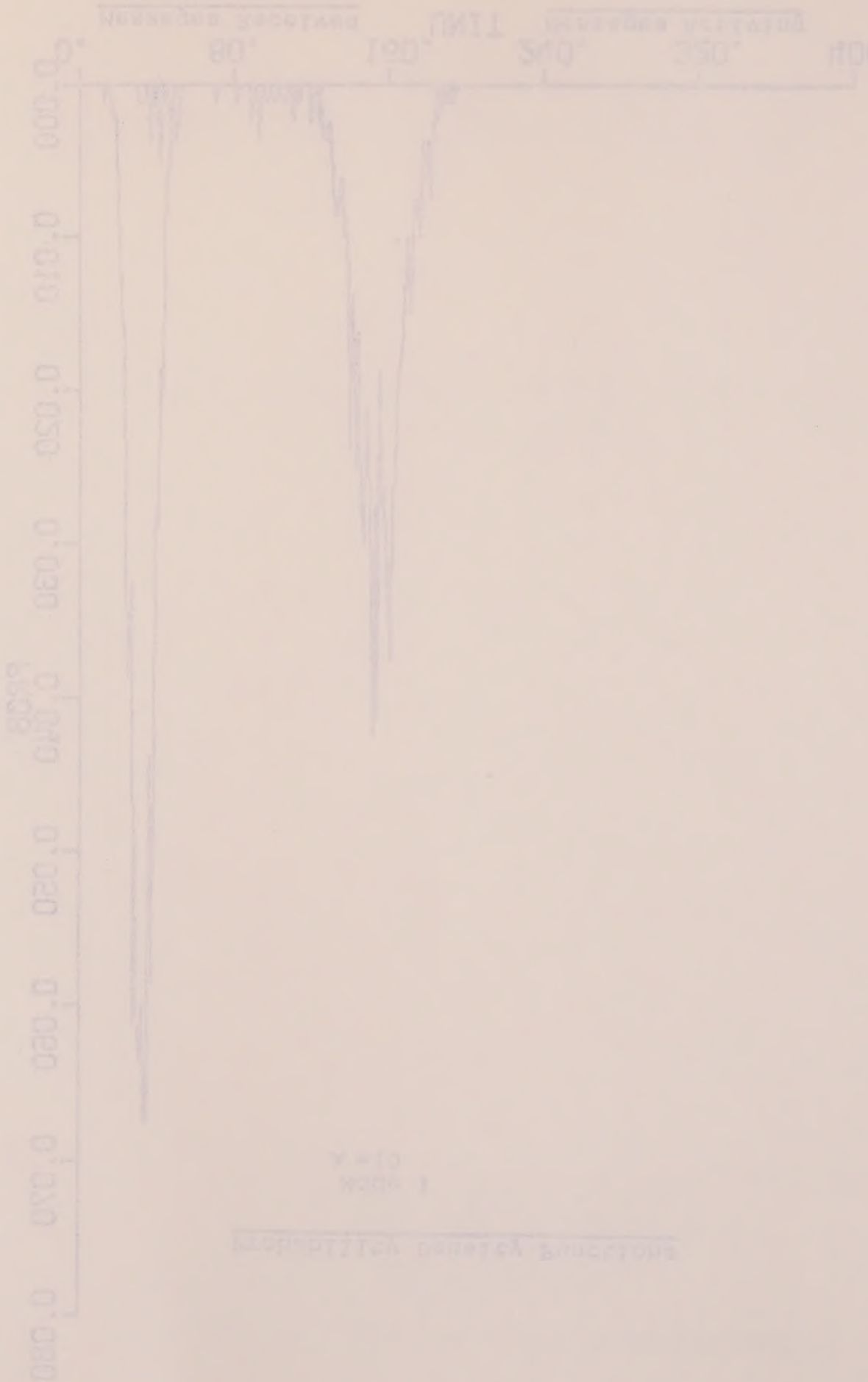
Mode-2
λ = 20
m = 100 slots



The number of arriving and received messages at the origin as a function of time.

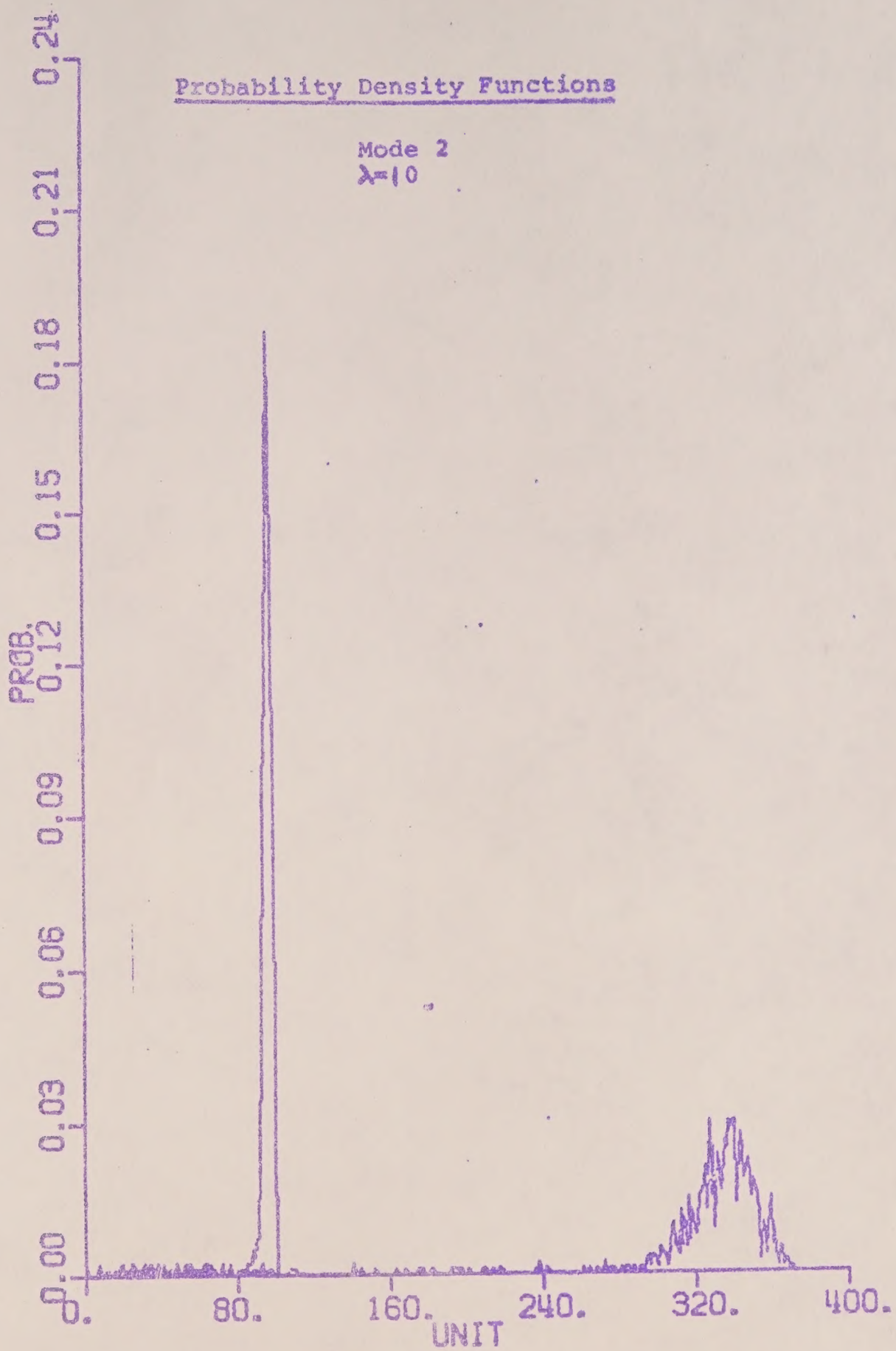


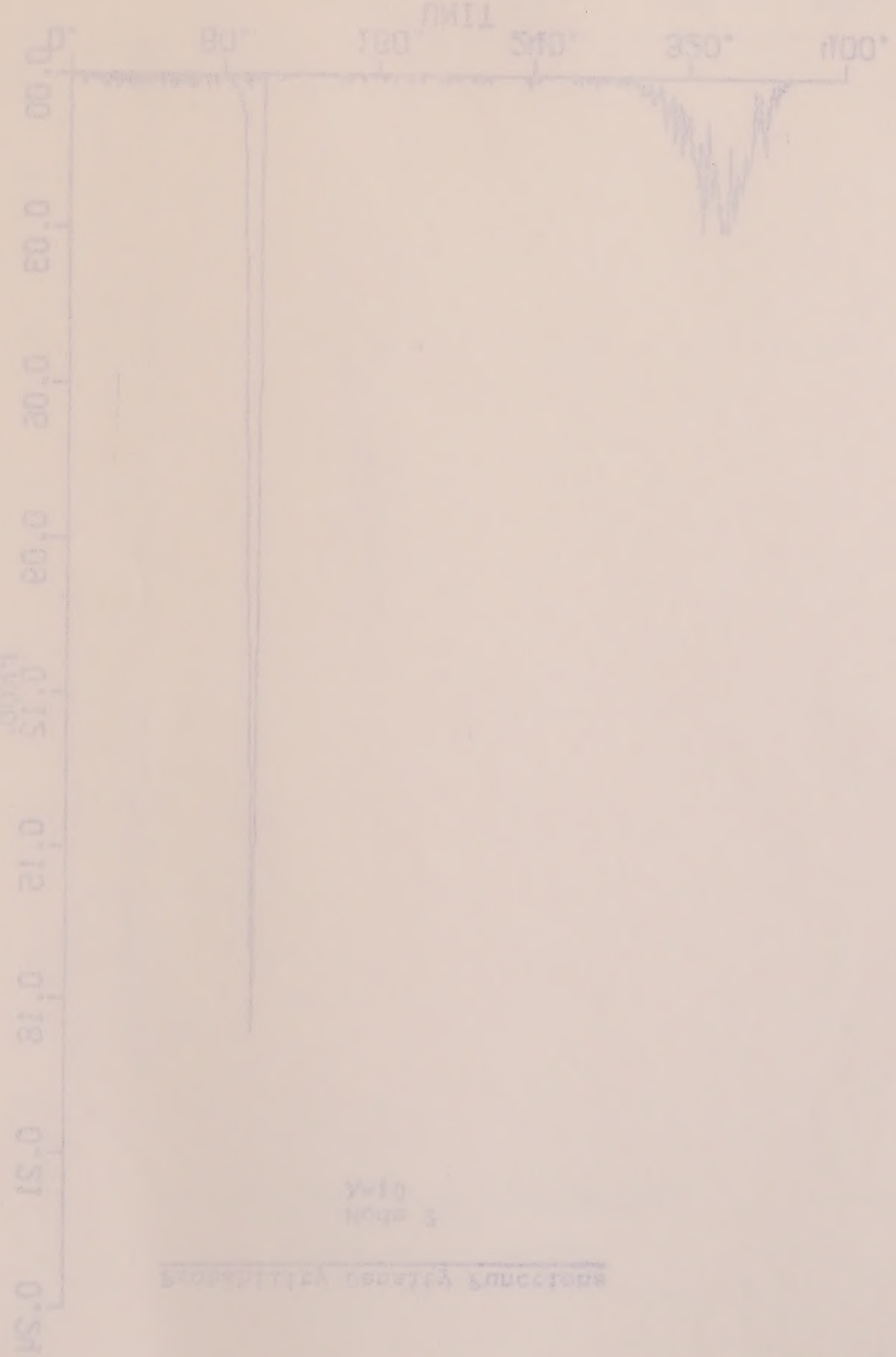

Probability Density Functions
Mode 1
λ = 10
PROB
0.000
0.010
0.020
0.030
0.040
0.050
0.060
0.070
0.080
0.
80.
160.
240.
320.
400.
Messages Received
UNIT
Messages Arriving



Probability Density Functions

Mode 2
$\lambda = 10$



Messages Received

Messages Arriving